JOURNAL OF DIFFERENTIAL EQUATIONS **157**, 120–149 (1999)
Article ID jdeq.1998.3623, available online at http://www.idealibrary.com on IDEAL®

The Inverted Pendulum: A Singularity Theory Approach

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Received May 11, 1998

The inverted pendulum with small parametric forcing is considered as an example of a wider class of parametrically forced Hamiltonian systems. The qualitative dynamics of the Poincaré map corresponding to the central periodic solution is studied via an approximating integrable normal form. At bifurcation points we

1. INTRODUCTION

The upper equilibrium of a pendulum can be stabilized by a vertical oscillation of the suspension point within a specific range of excitation frequencies and amplitudes. This follows from classical perturbation theory applied to the linearized equation of motion, e.g., see Van der Pol and Strutt [26], Stoker [29], and Hale [18]. A simple example is given by Arnold [1].

The corresponding bifurcation is determined by the non-linear dynamics. Our aim is to understand this dynamics in a qualitative way, with special interest in persistence. Here the symmetries of the system are first maintained. However, we also want to break these symmetries, and therefore consider a system that is slightly more general, but still in the $1\,\frac{1}{2}$-degree-of-freedom Hamiltonian setting.

We study the corresponding Poincaré map, following the approach of Broer and Vegter [12]. Normal form theory yields a planar Hamiltonian vector field which gives an integrable approximation of this map, valid for every angular displacement and small velocity of the pendulum. The relation between the Poincaré map and its approximation is briefly discussed in terms of perturbation theory.

At each bifurcation point of the approximating vector field a model is constructed that is locally equivalent to this approximation. We show structural stability of the local model by performing small perturbations

0022-0396/99 $30.00

that respect the symmetries, and conjugating these perturbations to the model by symmetry-preserving local morphisms. Here equivariant singularity theory is needed.

This paper focuses on the question of structural stability and is a follow-up of [7], where we computed the normal form and analyzed its bifurcations.

1.1. *Setting of the Problem*

The equation of motion of the inverted pendulum is given by

$$\ddot{x} = (\alpha + \beta\rho(t))\, V'(x), \tag{1}$$

where $V(x) = 1 - \cos x$, and $x \in \mathbb{S}^1$ is the deviation from the upper equilibrium $x = 0$. The forcing ρ is an arbitrary 2π-periodic C^∞ function. For simplicity, ρ is zero in average. Then, $\sqrt{\alpha}$ denotes the ratio of the "eigen-frequency" of the pendulum and the forcing frequency, while β controls the ratio of the forcing amplitude and the length of the pendulum.

Stability of the upper equilibrium is determined by the linearized equation

$$\ddot{x} - (\alpha + \beta\rho(t))\, x = 0.$$

For the Mathieu case with $\rho(t) = \cos t$ the stability diagram is well-known, see Meixner and Schäfke [22a], Van der Pol and Strutt [26], and Stoker [29]; also see Fig. 1. Van der Pol and Strutt [26] also treat the case

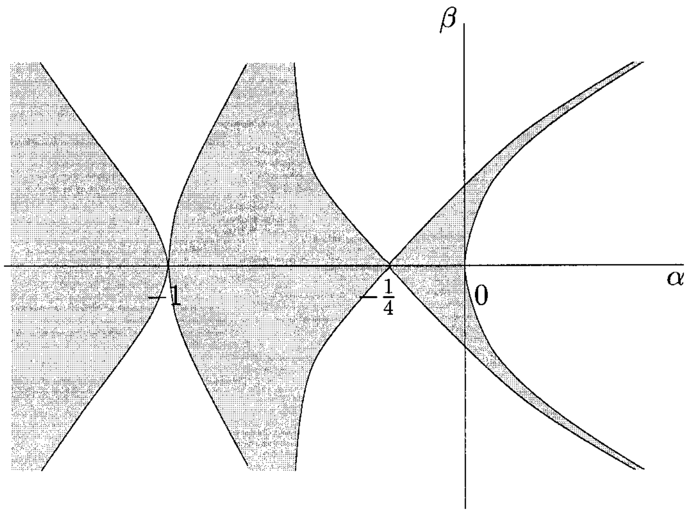

FIG. 1. Stability diagram for Mathieu's equation. Shaded regions correspond to parameter values for which the upper equilibrium is stable. We are interested in a small neighborhood of the origin of the parameter plane.

$\rho(t) = \operatorname{sign} \cos t$; an explicit solution for this case is given by Arnold [1]. Recent topological results can be found in Levi [20], Broer and Levi [8], and Broer and Simó [10]. Hale [18] discusses the nonlinear system, focusing on the computation of stable periodic solutions. For the nonlinear system around the *lower* equilibrium see Broer and Vegter [12] and Norris [24].

The inverted pendulum has a spatial symmetry, see below. We want to be able to break this symmetry, and therefore we consider (1) in the broader setting where V is a more general 2π-periodic C^∞ function. Introducing $y = \dot{x}$, we write (1) as a vector field

$$X = X_{\alpha,\beta}(x, y, t) = \frac{\partial}{\partial t} + y\frac{\partial}{\partial x} + (\alpha + \beta\rho(t))\, V'(x)\frac{\partial}{\partial y}, \tag{2}$$

where $(x, y, t) \in \mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1$, with y small, but with x global in $\mathbb{S}^1$. The parameters $(\alpha, \beta) \in \mathbb{R}^2$ are small. This is a Hamiltonian vector field with Hamilton function

$$H(x, y, t; \alpha, \beta) = \tfrac{1}{2}y^2 - (\alpha + \beta\rho(t))\, V(x). \tag{3}$$

If V is even, then X is equivariant with respect to the *spatial symmetry* $\mathscr{S}: (x, y, t) \mapsto (-x, -y, t)$. This means that $\mathscr{S}_* X = X$, and X is called $\mathscr{S}$-equivariant. If ρ is even, then X is time-reversible with respect to the *temporal symmetry* $\mathscr{R}: (x, y, t) \mapsto (x, -y, -t)$. This means that $\mathscr{R}_* X = -X$, and X is called $\mathscr{R}$-reversible. We note that the vector field of the inverted pendulum, i.e., $V(x) = 1 - \cos x$, is $\mathscr{S}$-equivariant.

For a system depending periodically on time it is natural to consider the Poincaré map[1] $P = P_{\alpha,\beta}: \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$, defined implicitly by

$$X^{2\pi}_{\alpha,\beta}(x, y, 0) = (P_{\alpha,\beta}(x, y), 2\pi),$$

where X^τ is the time-τ map of X. Since X is Hamiltonian, P is area-preserving. The symmetries of X are carried over as follows. Let $R, S: \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ be given by

$$R: (x, y) \mapsto (x, -y) \qquad \text{and} \qquad S: (x, y) \mapsto (-x, -y).$$

If X is $\mathscr{R}$-reversible, then P is R-reversible, i.e., $RPR = P^{-1}$. If X is $\mathscr{S}$-equivariant, then P is S-equivariant, i.e., $SPS = P$.

[1] Also called period map, return map, or stroboscopic map.

Remark 1. In the case of the inverted pendulum we have $V(x) = -V(x+\pi)$, giving rise to another symmetry. Indeed, take $\mathscr{T}: (x, y, t; \alpha, \beta) \mapsto (x+\pi, y, t; -\alpha, -\beta)$, then X is $\mathscr{T}$-equivariant as soon as $V(x) = -V(x+\pi)$. If X is $\mathscr{T}$-equivariant, then P is T-equivariant, where T: $(x, y; \alpha, \beta) \mapsto (x+\pi, y; -\alpha, -\beta)$. We use this symmetry to restrict the analysis of the dynamics of the inverted pendulum to the right half of the parameter plane, i.e., $\alpha \geqslant 0$.

1.2. *Experimental Results*

In this section we present some phase portraits of the Poincaré map P of the inverted pendulum, obtained by numerical integration using DsTool [2]. These phase portraits show how the dynamics of P depends on the symmetries of P.

Let us first consider the Mathieu case, where $\rho(t) = \cos t$. In this case P is S-equivariant and R-reversible. Near the origin the parameter plane falls apart in two regions, one region of stability and one of instability. These regions are displayed in the stability diagram at the bottom row of Fig. 2; Phase portraits of P in these regions are shown in the top row of this figure. They suggest that the stability boundary is a curve of pitchfork bifurcations.

We observe that both phase portraits are invariant under reflection in the horizontal axis and under rotation over angle π around the origin, due to the temporal and spatial symmetries of P, respectively.

We now break the temporal symmetry of P by taking $\rho(t) = \cos t + \sin 3t$. In this case, the graph of ρ has no reflection symmetry at all. The regions of stability and instability in a neighborhood of the origin $(\alpha, \beta) = (0, 0)$ of the parameter plane are of the same form as in the previous case. The center row of Fig. 2 shows phase portraits of P, for (α, β) in the stable and unstable regions. The temporal symmetry of P is broken now, but its spatial symmetry is not. Moreover, the phase portraits in this case resemble those in the Mathieu case; they are "qualitatively the same." In particular, the stability boundary is still a curve of pitchfork bifurcations. In Subsection 1.4 we shall see that the pitchfork bifurcation is destroyed if the spatial symmetry is broken.

1.3. *Towards a Theoretical Explanation*

The aim of this paper is to understand the "qualitative" dynamics of P, in particular its bifurcations, depending on the presence or absence of spatial and temporal symmetry. Further we want to study the persistence properties of this dynamics, in some restrictive sense. In this section we develop the theoretical framework needed to discuss these problems.

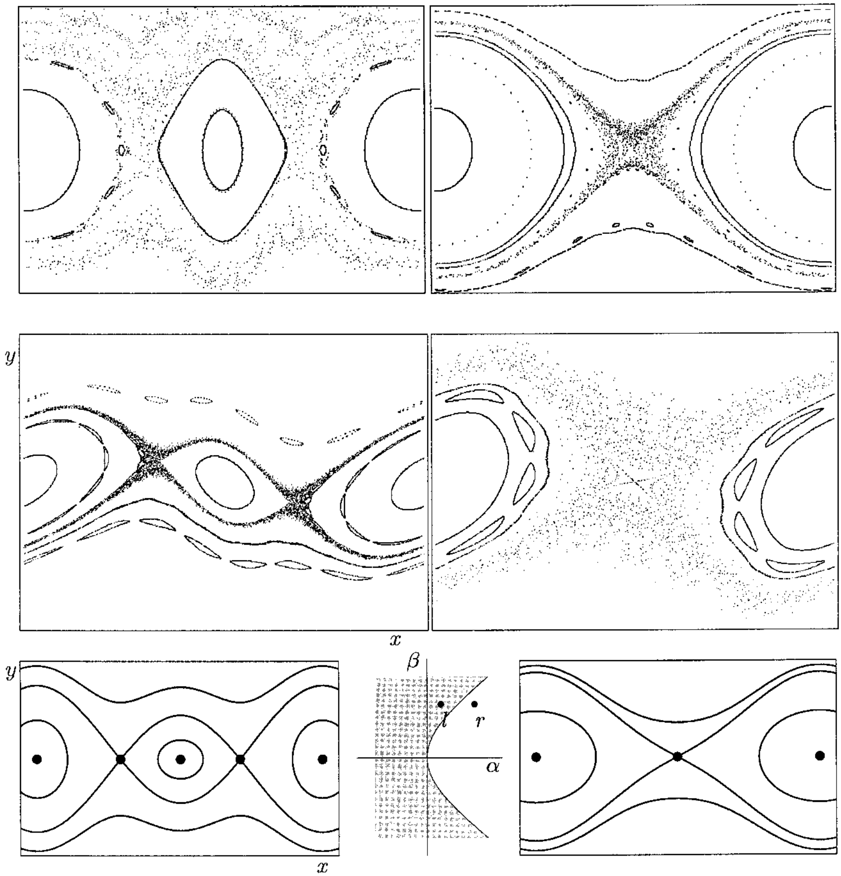

FIG. 2. Top, global Poincaré map of the inverted pendulum $V(x) = 1 - \cos x$, in the R-reversible setting with $\rho(t) = \cos t$, for two values of (α, β), indicated by l (left) and r (right) in the stability diagram at the bottom. Center, the same in the non-reversible case with $\rho(t) = \cos t + \sin 3t$. Bottom left and right, corresponding normal form phase portraits. Bottom center, local stability diagram, shading indicates stability of the upper equilibrium. The stability boundary is a line of Hamiltonian pitchfork bifurcations.

The Poincaré map P is too difficult to study in general, but it can be approximated by the Poincaré map of a planar vector field. We call the latter an *integrable* map, following Broer and Takens [11]. Indeed, up to a canonical transformation preserving the symmetries of P we can write

$$P \approx X_1^{2\pi},$$

for some Hamiltonian planar vector field X_1. The difference between P and its integrable approximation is infinitely flat as $(\alpha, \beta) \to (0, 0)$, for *global*

$x \in \mathbb{S}^1$ and small y. The vector field X_1 is computed by averaging out the time-dependence of X to arbitrarily high order in (α, β), preserving symmetries and the symplectic structure in the process.

All the qualitative dynamical features of $X_1^{2\pi}$ that persist under arbitrary flat perturbations are inherited by P. We give some brief remarks on this perturbation problem, and refer to [4, 5, 12] for more details.

Persistence of fixed points, their stability type, and bifurcations follows from the Implicit Function Theorem. However, all homoclinic and heteroclinic connections of $X_1^{2\pi}$ are expected to split transversally, creating a "chaotic sea," see Fig. 2. Finally, by KAM-theory certain invariant circles of $X_1^{2\pi}$ with Diophantine rotation number will survive, forming a Cantor foliation of positive measure. Other invariant circles can break apart, creating strings of "pendulum beads," again see Fig. 2. We note that this is a flat phenomenon and therefore hard to detect for small β, as is the "chaotic sea."

In the sequel only the integrable planar map $X_1^{2\pi}$ is considered. Its phase portrait is nothing but the collection of level curves of the corresponding Hamiltonian H_1, see also [12, 23]. Hence, stable and unstable critical points of H_1 correspond to fixed points of $X_1^{2\pi}$ of the same stability type, and bifurcations of H_1 match similar bifurcations of $X_1^{2\pi}$. Coinciding level curves of saddle points of H_1 correspond to homoclinic and heteroclinic connections of $X_1^{2\pi}$, and closed level curves of H_1 to invariant circles of $X_1^{2\pi}$. Thus we only need to consider the level curves of H_1, and therefore non-symplectic transformations—that change the time-parameterization, but not the orbit structure—are allowed from now on, cf. [4, 5, 7]. We apply such a transformation, as well as a rescaling, thereby simplifying H_1 to a form H_2. The Hamiltonian H_2 is of the form "kinetic energy plus potential energy." In particular, it is R-reversible, independent of the presence of temporal or spatial symmetry in the original system.

Of the dynamical features listed above, critical points, coinciding level curves, and closed level curves are easy to find. To analyze a local bifurcation, H_2 is considered in the *local* setting, that is, for (x, y, α, β) in a neighborhood of the bifurcation point. A *local model* for H_2 at a point $(x, y, \alpha, \beta) = (x_0, y_0, \alpha_0, \beta_0)$ is a family of Hamiltonians that has qualitatively the same dynamics as H_2 in a neighborhood of $(x_0, y_0, \alpha_0, \beta_0)$. Moreover the model has to be persistent under perturbations corresponding to small perturbations of V and ρ. For any such perturbation we show persistence by constructing a local morphism conjugating the perturbed system to the model. Here we use equivariant singularity theory.

The local model depends on the symmetries of X, and on whether or not the perturbations of V and ρ preserve these symmetries. We give such models in the symmetry contexts listed below. There are four cases in which the perturbations preserve the symmetries of X. In the last two cases, X is $\mathscr{S}$-equivariant, but its perturbations are not.

(1) The spatio-temporally symmetric case, where X and its perturbations are $\mathscr{R}$-reversible and $\mathscr{S}$-equivariant.

(2) The spatially symmetric case, where X and its perturbations are only $\mathscr{S}$-equivariant.

(3) The temporally symmetric case, where X and its perturbations are only $\mathscr{R}$-reversible.

(4) The non-symmetric case, where X and its perturbations are neither $\mathscr{S}$-equivariant nor $\mathscr{R}$-reversible.

(5) The perturbative temporally symmetric case, where X is $\mathscr{R}$-reversible and $\mathscr{S}$-equivariant, but its perturbations are only $\mathscr{R}$-reversible.

(6) The perturbative non-symmetric case, where X is $\mathscr{S}$-equivariant, but its perturbations are not.

Remark 2. The temporal symmetry of H_2 persists under arbitrary small perturbations of V and ρ. Indeed, the perturbed system is of the same form as H_2. Therefore the local models in the spatio-temporally and spatially symmetric cases coincide. The same holds for the local models in the cases without spatial symmetry, and the perturbative cases. Thus we need to consider only three cases.

Remark 3. There are three more symmetry contexts, where X is $\mathscr{R}$-reversible and optionally $\mathscr{S}$-equivariant, but its perturbations are not $\mathscr{R}$-reversible. Since H_2 is always R-reversible, these cannot be distinguished from the cases where the perturbations do not break the temporal symmetry, and therefore they are omitted.

We can retrieve the global dynamics of H_2 and prove their structural stability by constructing local models at singularities of H_2 and "gluing" them together, using a standard homotopy method. As an example we do so in the case of the pendulum, i.e., for $V(x) = 1 - \cos x$.

1.4. *The Dynamics of the Inverted Pendulum*

Using the method described in the previous section we find the global dynamics of the inverted pendulum, i.e., with $V(x) = 1 - \cos x$. The Poincaré map P of the inverted pendulum is S-equivariant and optionally R-reversible. As explained in Subsection 1.3, the dynamics of P qualitatively equals the dynamics of a planar Hamiltonian system with Hamilton function H_2. We discuss the dynamics of the planar system H_2, and comment briefly on the connection with the dynamics of P.

The phase portraits of H_2 are displayed at the bottom row of Fig. 2. At the point $(x, y) = (\pm\pi, 0)$ the Hamiltonian H_2 has a stable critical point for all small parameter values, corresponding to the stable lower equilibrium of the pendulum. At $(x, y) = (0, 0)$ there is a saddle point for (α, β) to the right of the curve $\alpha = \beta^2 + O(\beta^3)$, compare the stability diagram in Figs 2 or 3. This is a curve of pitchfork bifurcations; for (α, β) to its left the origin has become stable, with an unstable critical point on each side. The two saddles are in heteroclinic connection because of the spatial symmetry. The dynamics of H_2 is persistent under arbitrary small perturbations of ρ and small perturbations of V that preserve the spatial symmetry.

The connection with the dynamics of P is as explained in Subsection 1.3, with the following restriction. In the reduction to a planar system, in Subsection 2.1, we rescale the parameters (α, β) to remove a degenerate singularity at $(\alpha, \beta) = (0, 0)$. (Indeed, for $(\alpha, \beta) = (0, 0)$ every point on the x-axis is a fixed point of P, showing its degeneracy.) The rescaling is well-defined for $\beta \neq 0$. Therefore the relation between the dynamics of P and the dynamics of H_2 only holds for small (α, β) in the complement of the α-axis.[2]

Figure 2 shows that critical points and bifurcations of H_2 correspond to fixed points and bifurcations of P. The heteroclinic connections of H_2 correspond to heteroclinic tangle of P, and some of the closed level curves of H_2 correspond to strings of "pendulum beads" of P, while others persist.

The dynamics of H_2 is not persistent under perturbations that break the spatial symmetry. Under such a perturbation, the curve of pitchfork bifurcations falls apart into a curve of transcritical bifurcations and a curve of Hamiltonian saddle-node bifurcations. The heteroclinic connection of the unperturbed pendulum becomes a codimension-1 heteroclinic bifurcation in the absence of spatial symmetry, compare Fig. 3.

1.5. *A Brief Outline*

Let us give a short overview of the remainder of this paper. In the next section the local dynamics of the planar normal form is analyzed via local models. In Subsection 2.1 we first obtain the planar normal form H_1 of H, and transform H_1 to H_2. Some singularity theory, needed to construct local models of H_2, is discussed in Subsection 2.2. The local models, in each of the six symmetry contexts mentioned in Subsection 1.3, are presented in Subsection 2.3.

In Section 3 we discuss the dynamics of the inverted pendulum in the *global* setting, using the local models. Finally, Section 4 contains the proofs of the theorems of Subsection 2.3 that state the local models.

[2] On the α-axis P is simply the Poincaré map of the pendulum without forcing, with well-known integrable dynamics.

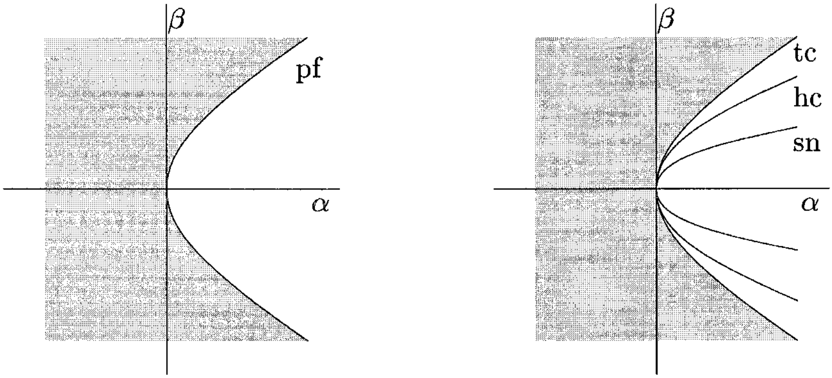

FIG. 3. Local stability diagram of the inverted pendulum in the spatially symmetric cases (left), and of a perturbed system that is not spatially symmetric (right). Stability of the origin (upper equilibrium) again is indicated by shading. The coding in this figure and others is as follows: **hc**, heteroclinic bifurcation; **pf**, (Hamiltonian) pitchfork bifurcation; **sn**, (Hamiltonian) saddle-node bifurcation; **tc**, transcritical bifurcation.

2. LOCAL MODELS FOR THE PLANAR SYSTEM

2.1. *Reduction to a Planar Hamiltonian System*

In this section we approximate the Poincaré map P of (2) by the flow over time 2π of a planar vector field X_1, as announced in Subsection 1.3. The vector field X_1 is a normal form of X, obtained by averaging out the time dependence of X to arbitrary high order in (α, β). Details of the normalization can be found in [7], compare also [12]. Since the normal form transformation is symplectic, X_1 is a Hamiltonian vector field with Hamiltonian H_1. In the second part of this section a non-symplectic transformation and a rescaling take H_1 to "kinetic plus potential energy form."

Let $H(x, y, t; \alpha, \beta) = \frac{1}{2}y^2 - (\alpha + \beta\rho(t))\, V(x)$ be the Hamiltonian of X, as before. For computational reasons we assume that the forcing function $\rho(t) = \sum_{k \in \mathbb{Z}\setminus\{0\}} a_k e^{ikt}$ satisfies

$$\sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{|a_k|^2}{k^2} = 1, \tag{4}$$

that is, the L^2-norm of its anti-derivative is 1.

THEOREM 4. *In the above circumstances, under the condition* (4), *there exists a symplectic time-preserving* C^∞ *near-identity diffeomorphism*

$\Psi\colon \mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1$, such that $\Psi_* X = X_1 + p_1$, where X_1 is a planar Hamiltonian vector field, given by the Hamilton function

$$H_1(x, y; \alpha, \beta) = \tfrac{1}{2} y^2 + O(|\alpha, \beta|^2\, y^2) + U(x; \alpha, \beta) + O(|\alpha, \beta|^3)$$
$$U(x; \alpha, \beta) = \tfrac{1}{2} \beta^2 (V'(x))^2 - \alpha V(x),$$

and $p_1 = p_1(x, y, t; \alpha, \beta)$ is a time-dependent vector field, flat in (α, β). If X is $\mathscr{R}$-reversible, then Ψ is $\mathscr{R}$-equivariant and H_1 is R-reversible. If X is $\mathscr{S}$-equivariant, then Ψ is $\mathscr{S}$-equivariant and H_1 is S-equivariant. If X is $\mathscr{T}$-equivariant, then Ψ is $\mathscr{T}$-equivariant and H_1 is T-equivariant. The remainder term $O(|\alpha, \beta|^2\, y^2)$ is independent of x, and the term $O(|\alpha, \beta|^3)$ is independent of y.

A proof of this theorem can be found in [7], and is based on standard normal form theory, see, e.g., [3, 11–13, 17, 19, 22, 27].

We further simplify H_1 by removing the remainder $O(|\alpha, \beta|^2\, y^2)$, using a non-symplectic transformation, which is allowed in one degree of freedom, see [4, 5, 7]. Indeed, let $\Psi\colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ be an arbitrary transformation, and X_2 the vector field corresponding to the transformed Hamiltonian $H_2 = H_1 \circ \Psi$, then

$$\Psi_* X_1 = (\det D\Phi)\, X_2.$$

The scalar $\det D\Phi$ corresponds to a rescaling of time, and is irrelevant for our qualitative purposes. Indeed, we are only interested in the level curves of H_1.

THEOREM 5. *Under the conditions of Theorem* 4, *there exists a local C^∞ near-identity diffeomorphism $\Phi\colon \mathbb{R} \to \mathbb{R}$, such that*

$$H_1(x, \Phi(y); \alpha, \beta) = H_2(x, y; \alpha, \beta),$$

where

$$H_2(x, y; \alpha, \beta) = \tfrac{1}{2} y^2 + U(x; \alpha, \beta) + O(|\alpha, \beta|^3).$$

The remainder is as before. If H_1 is R-reversible, then $\Phi(-y) = -\Phi(y)$ and H_2 is R-reversible. If H_1 is S-equivariant, then H_2 is also S-equivariant. If H_1 is T-equivariant, then H_2 is also T-equivariant.

Remark 6. The Hamiltonian H_2 is always R-reversible, hence its qualitative dynamics does not depend on the presence of R-reversibility in the original system H. However, if H is not R-reversible to start with, then at least one non R-equivariant transformation (like Φ) has to be allowed.

To overcome the degeneracy of H_2 at $(\alpha, \beta) = (0, 0)$ we also perform a rescaling, again cf. [7]. Indeed, we introduce new variables $\bar{x}$, $\bar{y}$ and new parameters $\bar{\alpha}$, $\bar{\beta}$ by the following relations:

$$\alpha = \bar{\beta}^2 \bar{\alpha}, \qquad \beta = \bar{\beta}, \quad x = \bar{x}, \quad \text{and} \quad y = |\bar{\beta}|\, \bar{y}.$$

Here $\beta = \bar{\beta}$ is considered small, but $\bar{\alpha}$ is not. This is a well-defined rescaling if we delete the α-axis from the parameter space. Since all singularities of H_2 are lying on the x-axis, the rescaling does not push any singularities to infinity, compare [12].

LEMMA 7 (Scaling). *In the above, let* $\bar{H}_2(\bar{x}, \bar{y}; \bar{\alpha}, \bar{\beta}) = \beta^{-2} H_2(x, y; \alpha, \beta)$. *Then*

$$\bar{H}_2(\bar{x}, \bar{y}; \bar{\alpha}, \bar{\beta}) = \tfrac{1}{2} \bar{y}^2 + \bar{U}(\bar{x}; \bar{\alpha}) + O(\bar{\beta}),$$

where

$$\bar{U}(\bar{x}; \bar{\alpha}) = \tfrac{1}{2}(V'(\bar{x}))^2 - \bar{\alpha} V(\bar{x}).$$

The remainder term $O(\bar{\beta})$ *is independent of* $\bar{y}$, *but depends on* $\bar{x}$. *If* H_2 *is R-reversible, then* $\bar{H}_2$ *is R-reversible. If* H_2 *is S-equivariant, then* $\bar{H}_2$ *is S-equivariant. If* H_2 *is T-equivariant, then* $\bar{H}_2$ *is T-equivariant.*

To simplify notation, from now on we omit all bars.

2.2. *Singularity Theory Leads to Local Models*

In this section we introduce some concepts of singularity theory, in particular the notion of a local model. General references for singularity theory are Martinet [21], discussing the standard (non-equivariant) theory, and Poénaru [25], who treats the general equivariant theory, also see Broer *et al.* [6]. Gibson [14] and Golubitsky and Schaeffer [15] provide a more geometric approach. Applications can be found in, e.g., Broer *et al.* [9], and Broer and Vegter [12]. Meyer [23] discusses bifurcations in Hamiltonian systems of codimension one and two, using singularity theory.

A germ $\bar{H}(x, y; p)$ at $(x, y; p) = 0$, with parameters $p = (p_1, ..., p_k)$, is called a *deformation* of a germ $h(x, y)$ if $\bar{H}(x, y; 0) = h(x, y)$. This deformation is *versal* if for any other deformation $G(x, y; q)$ of h, where $q = (q_1, ..., q_n)$, there exists a local reparameterization $\lambda: (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ and a local parameter-dependent diffeomorphism $\phi_q: \mathbb{R}^2 \to \mathbb{R}^2$ at 0, such that

$$G(x, y; q) = \bar{H}(\phi_q(x, y); \lambda(q)),$$

with $\phi_0 = \mathrm{id}$.

A germ $\bar{H}(x, y; p)$ at $(x, y; p) = 0$ is a *local model* of H_2 at $(x, y; \alpha, \beta) = (x_0, y_0; \alpha_0, \beta_0)$ if $H_2(x, y; \alpha_0, \beta_0)$ is locally, i.e., in a neighborhood of $(x, y) = (x_0, y_0)$, equivalent to $\bar{H}(x, y; 0)$, and moreover, $\bar{H}(x, y; p)$ is versal in the restricted class of deformations G that correspond to small perturbations of ρ and V that are allowed in the present symmetry context. Such perturbations constitute a class of "allowed" perturbations of H_2 that may be smaller than the class of all perturbations that fulfill the symmetry requirements. Indeed, this applies in one of the cases without spatial symmetry and in the perturbative cases, see Subsections 2.3.2 and 2.3.3, respectively. In other cases we can show that $\bar{H}(x, y; p)$ is a versal deformation in the class of all deformations that preserve the symmetries of H_2. Clearly, this is a sufficient condition for $\bar{H}(x, y; p)$ to be a local model.

A local model $\bar{H}$ of H_2 shows all local dynamics of H_2. Vice versa, we want H_2 to have all the dynamics of $\bar{H}$, that is, all perturbations $\bar{H}(x, y; p) - \bar{H}(x, y; 0)$ of $\bar{H}(x, y; 0)$ have to correspond to perturbations of V and ρ. A sufficient condition for this is that H_2 is a local model itself. In Subsection 2.3 we show that this does hold, except in the perturbative cases, i.e., where H_2 is spatially symmetric but its perturbations are not. In the latter cases we show that a specific, given family of perturbed systems H_2^ε of H_2 is a local model.

2.3. *Local Models for the Planar Hamiltonian System*

We now list the local models of H_2 in each of the six symmetry contexts. By remark 6, H_2 is always temporally symmetric, and hence its local dynamics does not depend on the presence or absence of temporal symmetry in the original Hamiltonian H, given by (3). Since all "allowed" perturbations of H_2 come from perturbations of ρ and V, they are of the same form as H_2 and hence they will also be temporally symmetric.

We conclude that the local models do not depend on the presence or absence of temporal symmetry in the original system. Thus the two cases with spatial symmetry can be treated simultaneously, and the same holds for the cases without spatial symmetry, and for the perturbative cases. This leaves us with three different symmetry contexts, to be discussed in the following three subsections.

2.3.1. *Local Models in the Cases with Spatial Symmetry*

In the spatio-temporally symmetric and spatially symmetric cases the Hamiltonian H_2 and its allowed perturbations are S-equivariant and R-reversible. Because of the spatial symmetry H_2 has a singularity at the origin $(x, y) = (0, 0)$, for all $(\alpha, \beta) \in \mathbb{R}^2$. The Taylor series of H_2 around $(x, y) = (0, 0)$, given by

$$\begin{aligned} H_2(x, y; \alpha, \beta) = {} & \tfrac{1}{2} y^2 + \tfrac{1}{2} V''(0)(V''(0) - \alpha)\, x^2 + \tfrac{1}{8} V''(0)\, V^{(4)}(0)\, x^4 \\ & + O(x^6, (\alpha - V''(0))\, x^4, \beta), \end{aligned} \tag{5}$$

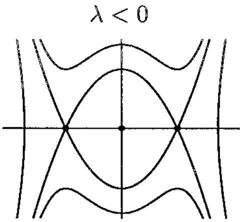

FIG. 4. Phase portraits of the local model $H_{RS} = \frac{1}{2}y^2 - \lambda x^2 + \sigma x^4$ in the cases with spatial symmetry, for $\sigma = -1$, showing a pitchfork bifurcation; see Theorem 9.

shows that the singularity at the origin undergoes a bifurcation for $\alpha = \alpha_0 := V''(0)$ and $\beta = 0$.

Remark 8. Of course H_2 can have other bifurcation points $(x, y) \neq (0, 0)$, but since we consider local bifurcations the spatial symmetry does not play a role there. Such bifurcation points therefore belong to the case without spatial symmetry, treated in Subsection 2.3.2.

The following theorem gives the local models in the spatio-temporally symmetric and spatially symmetric cases. The corresponding dynamics is shown in Figs. 4 and 5.

THEOREM 9 (Local Models in the Cases with Spatial Symmetry). *Let $H_2(x, y; \alpha, \beta) = \frac{1}{2}y^2 + U(x; \alpha) + O(\beta)$ as before, and let $(x, y; \alpha, \beta) = (0, 0; \alpha_0, 0)$ be a bifurcation point of H_2. In the cases with spatial symmetry, a local model for H_2 in the neighborhood of the bifurcation point is the equivariant cusp catastrophe, given by*

$$H_{RS}(x, y; \lambda) = \tfrac{1}{2}y^2 - \lambda x^2 + \sigma x^4,$$

under the generic conditions that $V''(0) \neq 0$ and $V^{(4)}(0) \neq 0$. Here $\sigma = \mathrm{sign}\,(V''(0)\, V^{(4)}(0))$. The two-parameter family H_2 itself is a local model.

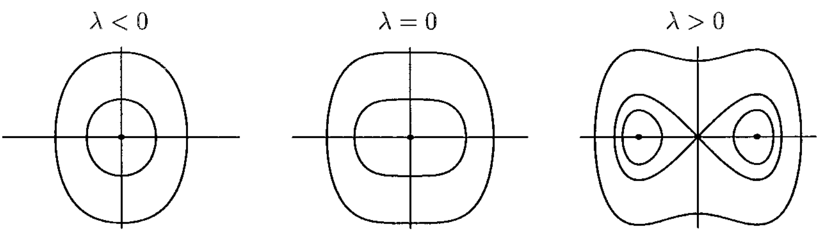

FIG. 5. Phase portraits of the local model $H_{RS} = \frac{1}{2}y^2 - \lambda x^2 + \sigma x^4$ in the cases with spatial symmetry, for $\sigma = +1$, showing a pitchfork bifurcation; see Theorem 9.

A proof is given in Section 4. In fact we prove the stronger statement that H_{RS} and H_2 are versal in the class of deformations that are even in x.

The conditions on V stated in the theorem have to be invariant under coordinate transformations $x \mapsto \phi(x)$, for any ϕ that preserves the spatial symmetry. To prove this we translate these to conditions on U. By the Taylor series (5) above, the conditions $V''(0) \neq 0$ and $V^{(4)}(0) \neq 0$ are equivalent to

$$\frac{\partial}{\partial \alpha} U''(0; \alpha_0) \neq 0, \qquad \text{and} \qquad U^{(4)}(0; \alpha_0) \neq 0.$$

These two conditions are generic. Indeed, a straightforward computation shows that the class of functions U of the form $U(x; \alpha) = \frac{1}{2}(V'(x))^2 - \alpha V(x)$ contains an open and dense subset of functions that satisfy the above conditions.

Furthermore,

$$U''(0; \alpha_0) = 0, \qquad U'(0; \alpha_0) = 0, \quad \text{and} \quad U^{(3)}(0; \alpha_0) = 0,$$

because of the spatial symmetry and $\alpha_0 = V''(0)$. These five conditions are invariant under coordinate transformations. In fact they are exactly the conditions of an equivariant pitchfork bifurcation; see, e.g., Guckenheimer and Holmes [16, p. 150].

We observe that H_{RS} models an equivariant pitchfork bifurcation. If $\sigma = -1$, then for negative λ it has a stable singularity at the origin surrounded by two saddles with coinciding level curves. These singularities coincide at the origin for $\lambda = 0$, leaving only a saddle at the origin for $\lambda \geqslant 0$. If $\sigma = +1$, then H_{RS} has a stable singularity at the origin for $\lambda \leqslant 0$, becoming unstable for $\lambda > 0$, while two stable singularities branch off. Figures 4 and 5 depict the phase portraits for $\sigma = -1$ and $\sigma = +1$, respectively.

2.3.2. *Local Models in the Cases Without Spatial Symmetry*

Now we turn to the temporally symmetric and non-symmetric cases where the Hamiltonian H_2 and its allowed perturbations are R-reversible only.

To locate the bifurcation points of H_2 we first find its singularities. Since $H_2(x, y; \alpha, \beta) = \frac{1}{2}y^2 + U(x; \alpha) + O(\beta)$, as in Lemma 7, we consider the singularities of $U(x; \alpha)$. Since $U'(x; \alpha) = V'(x)(V''(x) - \alpha)$, the singularities of U lie on curves in the (α, x)-plane given by $V'(x) = 0$ or $V''(x) = \alpha$. Bifurcations occur when the curve $V''(x) = \alpha$ intersects one of the straight lines $V'(x) = 0$, or when it folds. That is, $(x, y; \alpha, \beta) = (x_0, 0; \alpha_0, 0)$ is a bifurcation point of H_2 if $V'(x_0) = 0$ or $V^{(3)}(x_0) = 0$, and $\alpha_0 = V''(x_0)$ in

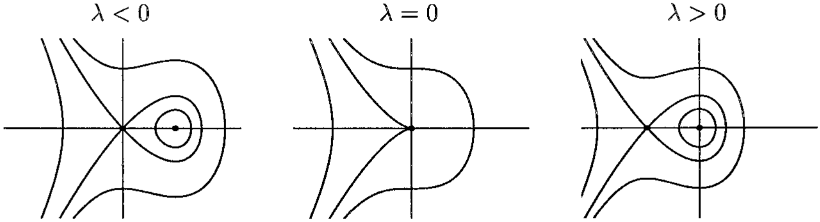

FIG. 6. Phase portraits of the local model $H_{R,1} = \frac{1}{2}y^2 + \lambda x^2 + x^3$ in the cases without spatial symmetry, showing a transcritical bifurcation; see Theorem 10(1).

both cases. Theorem 10, see below, states that in the first case H_2 has a transcritical bifurcation, and in the second case it has a saddle-node bifurcation, as one would expect.

Without loss of generality we assume that $x_0 = 0$. Indeed, any bifurcation point can be translated to the origin $(x, y) = (0, 0)$, and after such a translation the Hamiltonian H_2 is still of the same form as in Lemma 7.

By the above, a bifurcation occurs at $(x, y; \alpha, \beta) = (0, 0; \alpha_0, 0)$ if $V'(0) = 0$ or $V^{(3)}(0) = 0$, and $\alpha_0 = V''(0)$. This can also be seen from the Taylor series of H_2 around the bifurcation point, given by

$$H_2(x, y; \alpha, \beta) = \tfrac{1}{2}y^2 + V'(0)\,\tilde{\alpha}x + \tfrac{1}{2}(V''(0)\,\tilde{\alpha} + V'(0)\,V^{(3)}(0))\,x^2 + \tfrac{1}{6}(2V''(0)\,V^{(3)}(0) + V'(0)\,V^{(4)}(0))\,x^3 + O(x^4, \tilde{\alpha}x^3, \beta). \quad (6)$$

where $\tilde{\alpha} = V''(0) - \alpha$. The following theorem gives the local models in temporally symmetric and non-symmetric cases. The corresponding dynamics is shown in Figs. 6 and 7.

THEOREM 10 (Local Models in the Cases without Spatial Symmetry). *Let $H_2(x, y; \alpha, \beta) = \frac{1}{2}y^2 + U(x; \alpha) + O(\beta)$ as before, and let $(x, y; \alpha, \beta) = (0, 0; \alpha_0, 0)$ be a bifurcation point of H_2, with $\alpha_0 = V''(0)$. In the cases without spatial symmetry, there are two local models.*

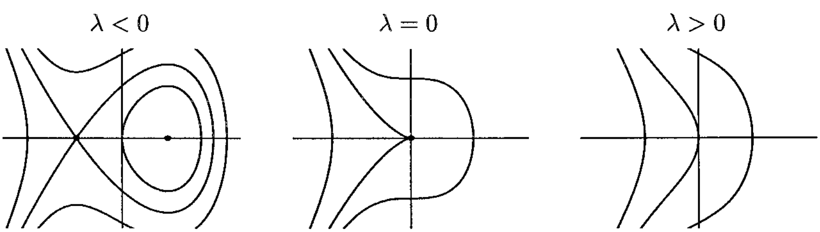

FIG. 7. Phase portraits of the local model $H_{R,2} = \frac{1}{2}y^2 + \lambda x + x^3$ in the cases without spatial symmetry, showing a Hamiltonian saddle-node bifurcation; see Theorem 10(2).

(1) *If* $V'(0)=0$, *then, under the generic conditions that* $V''(0)\neq 0$ *and* $V^{(3)}(0)\neq 0$, *a local model for* H_2 *in the neighborhood of the bifurcation point is the exchange catastrophe, given by*

$$H_{R,1}(x, y; \lambda)=\tfrac{1}{2}y^2+\lambda x^2+x^3.$$

(2) *If* $V^{(3)}(0)=0$, *then, under the generic conditions that* $V'(0)\neq 0$ *and* $V^{(4)}(0)\neq 0$, *a local model for* H_2 *in the neighborhood of the bifurcation point is the fold catastrophe, given by*

$$H_{R,2}(x, y; \lambda)=\tfrac{1}{2}y^2+\lambda x+x^3.$$

In both cases, the two-parameter family H_2 *itself is a local model.*

The proof is again postponed until Section 4. In the second case we actually prove the stronger statement that $H_{R,2}$ and H_2 are versal in the class of all deformations.

As before the conditions on V can be translated to conditions on U. In the first case of the theorem, the conditions $V'(0)=0$, $V''(0)\neq 0$ and $V^{(3)}(0)\neq 0$ are equivalent to

$$U'(0;\alpha)=0 \qquad \text{for all } \alpha, \qquad \frac{\partial}{\partial\alpha}U''(0;\alpha_0)\neq 0, \quad \text{and} \quad U^{(3)}(0;\alpha_0)\neq 0,$$

compare the Taylor series (6) above. The latter two of these conditions are generic in the class of functions of the form $U(x;\alpha)=\frac{1}{2}(V'(x))^2-\alpha V(x)$. Since $U''(0;\alpha_0)=0$, all three conditions are invariant under coordinate transformations. In fact they are the conditions of a transcritical bifurcation, again see Guckenheimer and Holmes [16, p. 150], and indeed, $H_{R,1}$ models such a bifurcation. It has two singularities, a center and a saddle, that coincide and exchange stability types at $\lambda=0$, see Fig. 6.

In the second case of the theorem, the conditions $V^{(3)}(0)=0$, $V'(0)\neq 0$, and $V^{(4)}(0)\neq 0$, together with $\alpha_0=V''(0)$, are equivalent to

$$U'(0;\alpha_0)=0, \qquad \frac{\partial}{\partial\alpha}U'(0;\alpha_0)\neq 0, \quad U''(0;\alpha_0)=0, \quad \text{and} \quad U^{(3)}(0;\alpha_0)\neq 0,$$

again see the Taylor series (6). These conditions are invariant under coordinate transformations; In fact they are the conditions of a saddle-node bifurcation, see Guckenheimer and Holmes [16, p. 148]. The second and fourth condition are generic. We observe that the model $H_{R,2}$ shows a Hamiltonian saddle-node bifurcation. Two singularities, a center and a saddle, exist for $\lambda<0$. They coincide at $\lambda=0$ and disappear for $\lambda>0$. Figure 7 displays the phase portraits.

2.3.3. *Local Models in the Perturbative Cases*

Finally we consider the perturbative cases, where the Hamiltonian H_2 is S-equivariant and R-reversible, but its perturbations are only R-reversible. These cases show the unfolding of the spatially symmetric model H_{RS} if perturbations that destroy the spatial symmetry are allowed. Because of its spatial symmetry the unperturbed Hamiltonian H_2 has a singularity at the origin $(x, y) = (0, 0)$. For $\beta = 0$ this singularity undergoes a pitchfork bifurcation at $\alpha = \alpha_0 := V''(0)$, see Subsection 2.3.1.

The Taylor series of the unperturbed Hamiltonian H_2 around $(x, y) = (0, 0)$ is given by (5). To get an idea of the local model in the perturbative cases, we compute the Taylor series of a perturbed Hamiltonian. Perturb the even "potential function" $V(x)$ of the original system (2) to $V^\varepsilon(x) = V(x) + \varepsilon W(x; \varepsilon)$, where W is some arbitrary smooth function, and ε is a small perturbation parameter. Let $H_2^\varepsilon(x, y; \alpha, \beta)$ be the corresponding planar Hamiltonian, given by

$$H_2^\varepsilon(x, y; \alpha, \beta) = \tfrac{1}{2} y^2 + U^\varepsilon(x; \alpha) + O(\beta),$$

where

$$U^\varepsilon(x; \alpha) = \tfrac{1}{2}((V^\varepsilon)'(x))^2 - \alpha V^\varepsilon(x),$$

compare Lemma 7. Here the remainder term $O(\beta)$ depends on $(x, \alpha, \beta, \varepsilon)$, but not on y. The following lemma shows that H_2^ε generically has a singularity near the origin $(x, y) = (0, 0)$ for all ε small.

LEMMA 11. *If $V''(0) \neq 0$, then H_2^ε has a critical point $(x, y) = (x_0(\alpha, \beta, \varepsilon), 0)$ for all $(\alpha, \beta) \in \mathbb{R}^2$ and ε sufficiently small. Here x_0 is a smooth function of the form*

$$x_0(\alpha, \beta, \varepsilon) = -\frac{W'(0; 0)}{V''(0)} \varepsilon + O(\varepsilon^2, \beta\varepsilon).$$

Proof. Application of the Implicit Function Theorem to the equation $(V^\varepsilon)'(x) = 0$ yields a smooth function $\tau: \mathbb{R} \to \mathbb{S}^1$ with $\tau(0) = 0$ such that $(V^\varepsilon)'(\tau(\varepsilon)) = 0$ for all ε sufficiently small. Hence the perturbed original vector field X^ε, given by

$$X^\varepsilon(x, y, t; \alpha, \beta) = \frac{\partial}{\partial t} + y \frac{\partial}{\partial x} + (\alpha + \beta\rho(t))(V^\varepsilon)'(x) \frac{\partial}{\partial y},$$

compare (2), has a stationary solution $(x, y, t) = (\tau(\varepsilon), 0, t)$ for all $(\alpha, \beta) \in \mathbb{R}^2$ and ε sufficiently small. As explained in Subsection 1.3, such a solution corresponds to a critical point of H_2^ε of the form $(x, y) = (x_0(\alpha, \beta, \varepsilon), 0)$, where $x_0(\alpha, \beta, \varepsilon) = O(\varepsilon)$. The explicit formula for x_0 follows by expanding the equation $(U^\varepsilon)'(x; \alpha) = 0$ in powers of ε. ▮

We compute the Taylor series of H_2^ε around $(x, y) = (x_0(\alpha, \beta, \varepsilon), 0)$. Abbreviate $\tilde{x} = x - x_0(\alpha, \beta, \varepsilon)$ and $\tilde{\alpha} = V''(0) + \varepsilon W''(0; 0) - \alpha$, then

$$H_2^\varepsilon(x, y; \alpha, \beta) = \tfrac{1}{2} y^2 + \tfrac{1}{2} V''(0)\, \tilde{\alpha} \tilde{x}^2 + \tfrac{1}{3} c \varepsilon \tilde{x}^3 + \tfrac{1}{8} V''(0)\, V^{(4)}(0)\, \tilde{x}^4 + O(\tilde{x}^6, \varepsilon \tilde{x}^4, \tilde{\alpha} \tilde{x}^4, \varepsilon^2 \tilde{x}^2, \varepsilon \tilde{\alpha} \tilde{x}^2, \beta \tilde{x}^2), \tag{7}$$

where

$$c = V''(0)\, W^{(3)}(0; 0) - V^{(4)}(0)\, W'(0; 0).$$

A local model for H_2 is given by the following theorem, its dynamics is shown in Figs. 8 and 9.

Theorem 12 (Local Model in the Perturbative Cases). *Let $H_2(x, y; \alpha, \beta) = \frac{1}{2} y^2 + U(x; \alpha) + O(\beta)$ as before, and let $(x, y; \alpha, \beta) = (0, 0; \alpha_0, 0)$ be a bifurcation point of H_2. Let $H_2^\varepsilon(x, y; \alpha, \beta)$ be a family of perturbed systems, defined as above. In the perturbative cases, a local model for H_2 in the neighborhood of the bifurcation point is given by*

$$H_{R*}(x, y; \lambda, \mu) = \tfrac{1}{2} y^2 - \lambda x^2 + \mu x^3 + \sigma x^4,$$

under the generic conditions that $V''(0) \neq 0$ and $V^{(4)}(0) \neq 0$. Here $\sigma = \operatorname{sign}(V''(0)\, V^{(4)}(0))$. If moreover $c = V''(0)\, W^{(3)}(0; 0) - V^{(4)}(0)\, W'(0; 0) \neq 0$, then the three-parameter family $H_2^\varepsilon(x, y; \alpha, \beta)$ is also a local model.

Again the proof is postponed until Section 4. Lemma 11 implies that

$$(U^\varepsilon)'(x_0(\alpha, 0, \varepsilon); \alpha) = 0 \qquad \text{for all } \alpha \text{ and small } \varepsilon,$$

and the conditions $V''(0) \neq 0$, $c \neq 0$ and $V^{(4)}(0) \neq 0$ are equivalent to the generic conditions

$$\frac{\partial}{\partial \alpha} (U^0)''(0; \alpha_0) \neq 0, \qquad \frac{\partial}{\partial \varepsilon} (U^0)'''(0; \alpha_0) \neq 0, \quad \text{and} \quad (U^0)^{(4)}(0; \alpha_0) \neq 0,$$

compare the Taylor series (7) above. The spatial symmetry for $\varepsilon = 0$ and the choice of α_0 imply that

$$(U^0)''(0; \alpha_0) = 0, \qquad (U^0)^{(3)}(0; \alpha_0) = 0.$$

These are the conditions for a cusp catastrophe in the present symmetry context, as modeled by H_{R*}, and they are invariant under coordinate transformations.

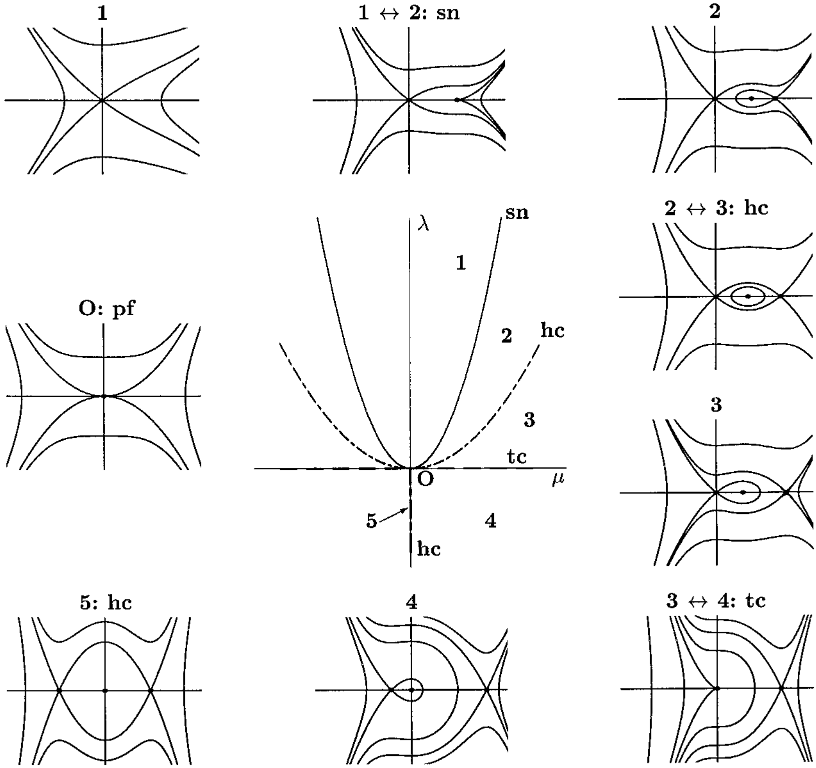

FIG. 8. Phase portraits of the local model $H_{R*} = \frac{1}{2}y^2 - \lambda x^2 + \mu x^3 + \sigma x^4$ in the perturbative cases with $\sigma = -1$, showing curves of Hamiltonian saddle-node, heteroclinic and transcritical bifurcations; see Theorem 12. Central, diagram in the (μ, λ)-parameter plane, organizing the phase portraits around. In reality the curves of saddle-node and transcritical bifurcations are closer together. For $\mu = 0$ we retrieve the case of spatial symmetry. The phase portraits for $\mu < 0$ are mirror images of those for $\mu > 0$ and are therefore omitted.

Let us describe the dynamics of H_{R*}. For $\mu = 0$ we retrieve H_{RS}. Now we suppose $\mu \neq 0$, and first let $\sigma = -1$. For negative λ the origin $(x, y) = (0, 0)$ is stable and surrounded by two saddle points. The phase portraits for $\mu < 0$ are mirror images of those for $\mu > 0$, hence there is a heteroclinic connection on the negative λ-axis. At $\lambda = 0$ the origin undergoes a transcritical bifurcation. For $0 < \lambda < \frac{1}{4}\mu^2$ it is a saddle, with a center and a saddle both on one side (to the right for $\mu > 0$, otherwise to the left). At $\lambda = \frac{1}{4}\mu^2$ there is a heteroclinic connection between the two saddles. Finally at $\lambda = \frac{9}{32}\mu^2$ a Hamiltonian saddle-node bifurcation occurs, leaving only the saddle in the origin for $\lambda > \frac{9}{32}\mu^2$. The phase portraits in this case are displayed in Fig. 8.

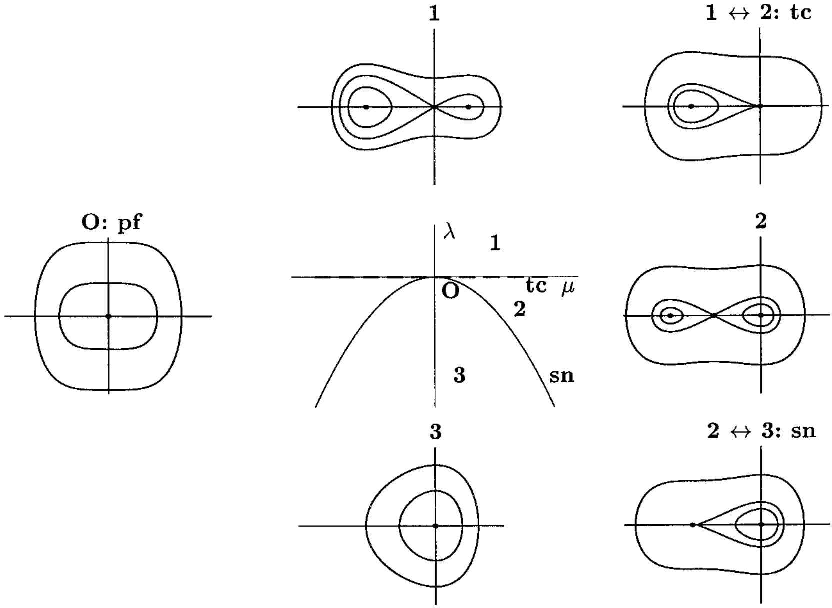

FIG. 9. Phase portraits of the local model $H_{R*} = \frac{1}{2}y^2 - \lambda x^2 + \mu x^3 + \sigma x^4$ in the perturbative cases with $\sigma = +1$, showing curves of Hamiltonian saddle-node and transcritical bifurcations; see Theorem 12. Central: diagram in the (μ, λ)-parameter plane, organizing the phase portraits around. For $\mu = 0$ we retrieve the case of spatial symmetry. The phase portraits for $\mu < 0$ are mirror images of those for $\mu > 0$, and are thus omitted.

Now let $\sigma = +1$. Again the phase portraits for $\mu < 0$ are mirror images of those for $\mu > 0$. For $\lambda < -\frac{9}{32}\mu^2$ there is only a center at the origin. At $\lambda = -\frac{9}{32}\mu^2$ a Hamiltonian saddle-node bifurcation occurs. For $-\frac{9}{32}\mu^2 < \lambda < 0$ the new singularities are to the right of the origin if $\mu < 0$, otherwise to the left. At $\lambda = 0$ there is a transcritical bifurcation at the origin, and for positive λ the origin is unstable and surrounded by two centers, see Fig. 9.

Remark 13. The curves of saddle-node and transcritical bifurcations in Figs. 8 and 9 have second order of contact. This can be understood as follows. Write $H_{R*}(x, y; \lambda, \mu) = \frac{1}{2}y^2 + U_{R*}(x; \lambda, \mu)$, with $U_{R*}(x; \lambda, \mu) = -\lambda x^2 + \mu x^3 + \sigma x^4$, and take $\sigma = -1$ for simplicity. Then U_{R*} is a deformation of $-x^4$ that is singular for $(x, \lambda, \mu) \in \{0\} \times \mathbb{R}^2$. We identify the parameter plane, i.e., the (λ, μ)-plane, of U_{R*} with this singular set. A versal deformation of $-x^4$ is the ordinary cusp catastrophe $\widetilde{U}(x; \eta, \theta) = -\eta x - \theta x^2 - x^4$. By definition of versality, $\widetilde{U} \circ \phi = U_{R*}$, for some map $\phi: \mathbb{R}^3 \to \mathbb{R}^3$ that takes fibers $\mathbb{R} \times \{(\lambda, \mu)\}$ to fibers $\mathbb{R} \times \{(\eta, \theta)\}$.

We discuss the image under the map ϕ of the parameter plane of U_{R*} and the bifurcation curves in this plane. One can show that ϕ maps the

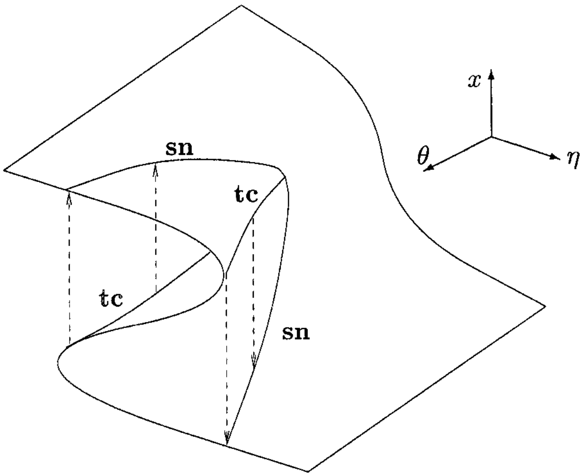

FIG. 10. The bifurcation curves of $H_{R*}(x, y; \lambda, \mu) = \frac{1}{2}y^2 - \lambda x^2 + \mu x^3 + \sigma x^4$ in the (λ, μ)-plane can be identified with curves in the singular set of the ordinary cusp catastrophe $\tilde{U}(x; \eta, \theta) = -\eta x - \theta x^2 - x^4$ (for $\sigma = -1$). The curve of transcritical bifurcations of H_{R*} corresponds to the fold curves of $\tilde{U}$, while the curve of saddle-node bifurcations of H_{R*} corresponds to the projection in the x-direction of the fold curves of $\tilde{U}$ onto the "free" leaf, as indicated.

parameter plane diffeomorphically to the singular set of $\tilde{U}$ in $\mathbb{R}^3$. In particular, it maps the line $(x, \lambda, \mu) = (0, 0, \mu)$ of degenerate points of U_{R*}—where the transcritical bifurcation of H_{R*} takes place—to the fold curves of $\tilde{U}$, as indicated in Fig. 10.

The curve of fold points of U_{R*}—where H_{R*} undergoes a saddle-node bifurcation—is of the form $(x, \lambda, \mu) = (x(\mu), \lambda(\mu), \mu)$. This curve is also mapped to the fold curves of $\tilde{U}$. But, since $x(\mu) \neq 0$ for $\mu \neq 0$, the image under ϕ of the fold curve in the *parameter* plane is $\phi(0, \lambda(\mu), \mu)$. This is the projection in the x-direction of the fold curves of $\tilde{U}$ onto the "free" leaf of the singular set of $\tilde{U}$, again see Fig. 10. One easily shows that the curves of fold points and exchange points (in the singular set of $\tilde{U}$) have second order contact at the cusp point $(x, \eta, \theta) = (0, 0, 0)$.

3. GLOBAL DYNAMICS OF THE INVERTED PENDULUM

In this section we discuss the case of the inverted pendulum, i.e., $V(x) = 1 - \cos x$, and determine the dynamics of H_2 for global $x \in \mathbb{S}^1$, using Theorem 9. The Hamiltonian H_2, given by

$$H_2(x, y; \alpha, \beta) = \tfrac{1}{2}y^2 + \tfrac{1}{2}\sin^2 x - \alpha(1 - \cos x) + O(\beta),$$

is T- and S-equivariant and R-reversible. Because of the first symmetry we restrict to the case $\alpha \geqslant 0$ without loss of generality.

The global dynamics of H_2 is summarized in the following theorem, also see Fig. 11 for a bifurcation diagram and global phase portraits.

THEOREM 14 (Dynamics of the Inverted Pendulum). *Let* $V(x) = 1 - \cos x$. *Then* $H_2(x, y; \alpha, \beta) = \frac{1}{2}y^2 + U(x; \alpha) + O(\beta)$, *with* $U(x; \alpha) = \frac{1}{2}\sin^2 x - \alpha(1 - \cos x)$, *is the planar normal form of the inverted pendulum. For all parameter values* (α, β) *there exists a center at* $(x, y) = (\pi, 0)$. *There is a singularity at the origin* $(x, y) = (0, 0)$; *its stability type changes at a curve in the parameter plane of the form* $\alpha = 1 + O(\beta)$. *This is a curve of pitchfork bifurcations. To its left the origin is stable and surrounded by two saddles in heteroclinic connection. To its right there is only a single saddle point at the origin. There are no other bifurcations. This system is structurally stable in the world of spatially symmetric systems.*

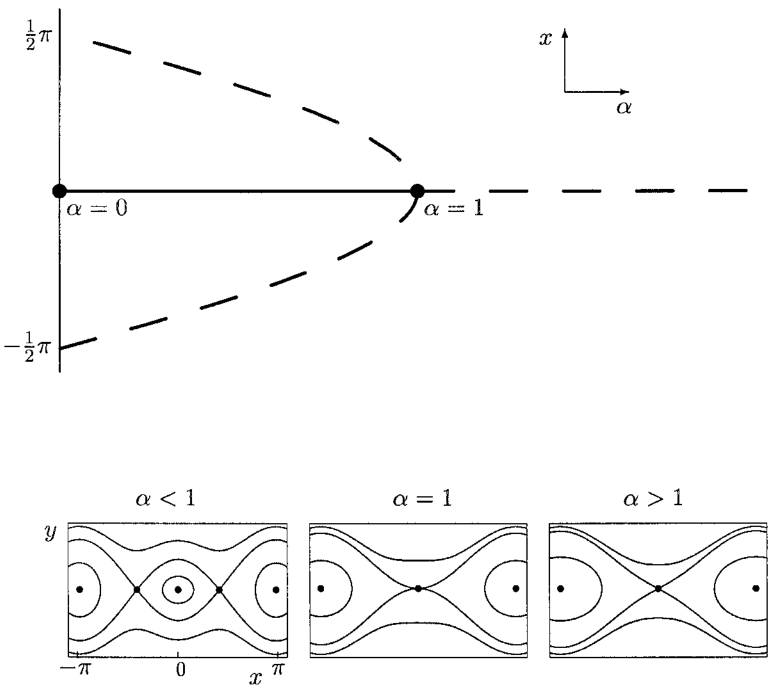

FIG. 11. The inverted pendulum for $x \in \mathbb{S}^1$; see Theorem 14. Top, bifurcation diagram in the (α, x)-plane for $\beta = 0$. Dashed lines indicate unstable equilibria. Bottom, corresponding global phase portraits. For $\alpha = 1$ a Hamiltonian pitchfork bifurcation occurs, stabilizing the origin $(x, y) = (0, 0)$.

Proof. By Theorem 9, H_2 is equivalent (after reparameterization) to H_{RS}, with $\sigma = -1$, in an open neighborhood of the bifurcation point $(x, y, \alpha, \beta) = (0, 0, 1, 0)$ in $\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}^2$. Hence the bifurcation is a pitchfork. The heteroclinic connections exist and persist because of the spatial symmetry.

We prove structural stability of H_2. Let H_2^ε be a family of perturbed systems in the spatially symmetric context, depending smoothly on a perturbation parameter ε. We have to show that there exist diffeomorphisms $\lambda^\varepsilon: \mathbb{R}^2 \to \mathbb{R}^2$ and $\Psi^{\alpha, \beta, \varepsilon}: \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$, such that

$$H_2^\varepsilon(x, y; \alpha, \beta) = H_2(\Psi^{\alpha, \beta, \varepsilon}(x, y); \lambda^\varepsilon(\alpha, \beta)), \tag{8}$$

for all (x, y) in a neighborhood I of $\mathbb{S}^1 \times \{0\}$ in $\mathbb{S}^1 \times \mathbb{R}$ and all $(\alpha, \beta, \varepsilon)$ in a neighborhood J of $\mathbb{R}_{>0} \times \{0\} \times \{0\}$ in $\mathbb{R}^3$.

The singularities of H_2 involved in the pitchfork bifurcation all lie on the x-axis and satisfy $x = 0$ or $\cos x = \alpha$. Hence they lie in a neighborhood $I_1 \subset I$ of $(-\frac{1}{2}\pi, \frac{1}{2}\pi) \times \{0\}$, for all $\alpha > 0$ and β sufficiently small. We first construct a diffeomorphism $\Psi^{\alpha, \beta, \varepsilon}$ on I_1.

By Theorem 9, H_2 is a local model at the bifurcation point $(x, y, \alpha, \beta) = (0, 0, 1, 0)$, hence there exist $\Psi^{\alpha, \beta, \varepsilon}$ and λ^ε of the above form such that (8) holds in an open neighborhood of $(x, y, \alpha, \beta, \varepsilon) = (0, 0, 1, 0, 0)$ in $\mathbb{S} \times \mathbb{R} \times \mathbb{R}^3$. Since the only singularities of H_2 in I_1 are those mentioned above, $\Psi^{\alpha, \beta, \varepsilon}$ and λ^ε can be chosen such that (8) holds for all $(x, y) \in I_1$ and all $(\alpha, \beta, \varepsilon) \in J$ for sufficiently small neighborhoods I_1 and J, compare [12]. In the sequel we denote this $\Psi^{\alpha, \beta, \varepsilon}$ by $\Psi_1^{\alpha, \beta, \varepsilon}$.

The singularity of H_2 at $(x, y) = (\pm\pi, 0)$ is non-degenerate and stable. One easily shows that a local model for H_2 at $(x, y, \alpha, \beta) = (\pm\pi, 0, \alpha, 0)$, for any $\alpha > 0$, is given by the Morse germ $\tilde{H}(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2$. Moreover, H_2 itself is also a local model. Hence (8) holds in an open neighborhood $I_2 \subset I \backslash I_1$ of $(x, y) = (\pm\pi, 0)$ for all $(\alpha, \beta, \varepsilon) \in J$, with diffeomorphisms λ^ε, $\Psi_2^{\alpha, \beta, \varepsilon}$. Observe that we can take the same λ^ε as in the previous case, since the local model does not require parameters.

The set $I \backslash (I_1 \cup I_2)$ contains only regular points of H_2, for all $\alpha > 0$ and β sufficiently small. Hence a standard homotopy method, compare [12], yields a diffeomorphism $\Psi^{\alpha, \beta, \varepsilon}$, equal to $\Psi_1^{\alpha, \beta, \varepsilon}$ on I_1, and equal to $\Psi_2^{\alpha, \beta, \varepsilon}$ on I_2, such that (8) holds with diffeomorphisms λ^ε, $\Psi^{\alpha, \beta, \varepsilon}$, for all $(x, y) \in I$, $(\alpha, \beta, \varepsilon) \in J$. ▮

Theorem 12 (with $\sigma = -1$) implies that, under a perturbation that destroys the spatial symmetry, the pitchfork bifurcation of the inverted pendulum falls apart in a transcritical, a heteroclinic and a saddle-node bifurcation:

THEOREM 15 (Dynamics of the Perturbed Inverted Pendulum). *Let $V^\varepsilon(x)=1-\cos x+\varepsilon W(x;\varepsilon)$, with W arbitrary. Then $H_2^\varepsilon(x, y; \alpha, \beta)=\frac{1}{2}y^2+U^\varepsilon(x;\alpha)+O(\beta)$, with $U^\varepsilon(x;\alpha)=\frac{1}{2}((V^\varepsilon)'(x))^2-\alpha V^\varepsilon(x)$, is the normal form of a perturbed inverted pendulum. For all parameter values (α, β) there exists a center at $(x, y)=(\pm\pi, 0)$, and a singularity at $O(\varepsilon)$ distance from the origin $(x, y)=(0, 0)$. The unperturbed system has a curve of pitchfork bifurcations in the parameter plane of the form $\alpha=1+O(\beta)$. If $c=V''(0)\, W^{(3)}(0; 0)-V^{(4)}(0)\, W'(0; 0)\neq 0$, then this curve falls apart in curves of transcritical, heteroclinic and saddle-node bifurcations of the form $\alpha=c_i+O(\beta)$, with $i=1, 2, 3$, respectively. All these bifurcations take place at $O(\varepsilon)$ distance from the origin $(x, y)=(0, 0)$, and the c_i satisfy $c_i=1+O(\varepsilon)$ and $c_1<c_2<c_3$. There are no other bifurcations. The family H_2 is structurally stable in the world of all perturbative cases.*

The proof is analogous to that of Theorem 14 and is therefore omitted.

Remark 16. In the original parameter plane (see Lemma 7) the pitchfork bifurcation of the inverted pendulum takes place at a curve of the form $\alpha=\beta^2+O(\beta^3)$. Under a perturbation destroying the spatial symmetry it falls apart in curves of transcritical, heteroclinic and saddle-node bifurcations of the form $\alpha=c_i\beta^2+O(\beta^3)$, with c_i as in Theorem 15. This explains Fig. 3.

Remark 17. The relation between the dynamics of H_2 or H_2^ε and the Poincaré map of the original system is as explained in Subsections 1.3 and 1.4; compare also Fig. 2.

4. PROOFS OF THEOREMS 9, 10, AND 12

In this section Theorems 9, 10, and 12 are proved. The proofs are based on the Malgrange–Mather Preparation Theorem and equivariant versions thereof, see the references on singularity theory given above. The various symmetry contexts can be treated almost the same. Therefore we discuss the spatially symmetric case extensively, and point out the differences with the other cases. In particular, by Remark 6, the cases with and without temporal symmetry can be treated as one.

4.1. *Proof of Theorem* 9

Let us start with Theorem 9, i.e., the cases with spatial symmetry, where H_2 and its perturbations are R-reversible and S-equivariant.

Proof of Theorem 9. Assume that $V''(0) \neq 0$ and $V^{(4)}(0) \neq 0$. Then we have to prove that a local model for H_2 around the bifurcation point $(x, y; \alpha, \beta) = (0, 0; V''(0), 0)$ is given by

$$H_{RS}(x, y; \lambda) = \tfrac{1}{2} y^2 - \lambda x^2 + \sigma x^4,$$

where $\sigma = \operatorname{sign} V''(0)\, V^{(4)}(0)$, and that H_2 itself is a local model.

The Taylor series of H_2 around $(x, y; \alpha, \beta) = (0, 0; V''(0), 0)$ is given by

$$H_2(x, y; \alpha, \beta) = \tfrac{1}{2} y^2 + \tfrac{1}{2} V''(0)(V''(0) - \alpha)\, x^2 + \tfrac{1}{8} V''(0)\, V^{(4)}(0)\, x^4 + O(x^6, (\alpha - V''(0))\, x^4, \beta),$$

compare (5). First a local diffeomorphism $\phi: (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and an invertible reparameterization $\alpha \mapsto \bar{\alpha}$ are constructed such that

$$H_2(\phi(x), y; \alpha, \beta) = \tfrac{1}{2} y^2 - \bar{\alpha}(\alpha)\, x^2 + \sigma x^4 + O(\bar{\alpha}(\alpha)\, x^4, \beta).$$

This is a deformation of $\frac{1}{2} y^2 + \sigma x^4$, just as H_{RS}. Then these deformations are shown to be versal in the class of deformations that preserve the spatial symmetry.

We abbreviate $d = \frac{1}{8} |V''(0)\, V^{(4)}(0)|$, then $d > 0$, and

$$H_2(x, y; \alpha, \beta) = \tfrac{1}{2} y^2 + \tfrac{1}{2} V''(0)(V''(0) - \alpha)\, x^2 + \sigma\, d x^4 + x^6 f(x^2) + O((\alpha - V''(0))\, x^4, \beta),$$

for some function f. Let $\phi^{-1}(x) = x \sqrt[2]{d + \sigma x^2 f(x^2)}$, and $\bar{\alpha} = \frac{1}{2} d^{-1/2} V''(0)(V''(0) - \alpha)$, then ϕ is well-defined for sufficiently small x and preserves the spatial symmetry, and

$$\bar{H}_2(x, y; \bar{\alpha}, \beta) := H_2(\phi(x), y; \alpha, \beta) = \tfrac{1}{2} y^2 + \bar{\alpha} x^2 + \sigma x^4 + O(\bar{\alpha} x^4, \beta).$$

This finishes the first step. To simplify notation we delete the bars from now on.

Observe that H_2 and H_{RS} are deformations of $h(x, y) := \frac{1}{2} y^2 + \sigma x^4$. We prove that they are versal in the class of deformations that are even in x. Let $\mathscr{E}_S$ denote the ring of germs at 0 of functions $f: (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ that are even in the first variable: $f(x, y) = f(-x, y)$. Let $\mathscr{M}_S$ be its maximal ideal, consisting of germs f with $f(0, 0) = 0$, and let $J(h) \subset \mathscr{E}_S$ be the Jacobian ideal of h, that is, the ideal generated by $(\partial/\partial x)\, h(x, y)$ and $(\partial/\partial y)\, h(x, y)$. Then

$$J(h) + \mathbb{R} \left\{ \frac{\partial}{\partial \lambda} H_{RS} \bigg|_{\lambda = 0} \right\} = \mathscr{M}_S$$

$$J(h) + \mathbb{R} \left\{ \frac{\partial}{\partial \alpha} H_2 \bigg|_{\alpha = \beta = 0}, \frac{\partial}{\partial \beta} H_2 \bigg|_{\alpha = \beta = 0} \right\} = \mathscr{M}_S.$$

According to the equivariant version of the Malgrange–Mather Preparation Theorem, see [25], this is a sufficient (and necessary) condition for H_{RS} and H_2 to be versal deformations. This completes the proof. ∎

4.2. *Proof of Theorem* 10

In this section Theorem 10, concerned with the cases without spatial symmetry, is proved. In these cases H_2 is R-reversible, but not S-equivariant We recall that there are two cases: at the bifurcation point $(x, y; \alpha, \beta) = (0, 0; V''(0), 0)$ either two curves of singularities in the (α, x)-plane, given by $x = 0$ and $\alpha = V''(x)$, intersect, forming an exchange, or a curve of singularities $\alpha = V''(x)$ folds. We start with the first case.

Proof of Theorem 10(1). Assume that $V'(0) = 0$, $V''(0) \neq 0$ and $V^{(3)}(0) \neq 0$. Then we have to prove that a local model for H_2 around the bifurcation point $(x, y; \alpha, \beta) = (0, 0; V''(0), 0)$ is given by

$$H_{R,1}(x, y; \lambda) = \tfrac{1}{2} y^2 + \lambda x^2 + x^3,$$

and that H_2 itself is a local model.

The Taylor series of H_2 around the bifurcation point is

$$\begin{aligned} H_2(x, y; \alpha, \beta) = \tfrac{1}{2} y^2 + \tfrac{1}{2} V''(0)(V''(0) - \alpha)\, x^2 + \tfrac{1}{3} V^{(3)}(0)\, V''(0)\, x^3 \\ + O((V''(0) - \alpha)\, x^3, x^4, \beta), \end{aligned}$$

compare (6). Analogous to the previous case, there exists a local diffeomorphism $\phi: (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and an invertible reparameterization $\alpha \mapsto \bar{\alpha}$ such that

$$\bar{H}_2(x, y; \bar{\alpha}, \beta) := H_2(\phi(x), y; \alpha, \beta) = \tfrac{1}{2} y^2 - \bar{\alpha}(\alpha)\, x^2 + x^3 + O(\bar{\alpha}(\alpha)\, x^3, \beta).$$

Again we delete the bars from now on. The Hamiltonians H_2 and $H_{R,1}$ are deformations of $h(x, y) := \tfrac{1}{2} y^2 + x^3$.

Observe that $H_2|_{\beta=0}$ has a singularity at the origin $(x, y) = (0, 0)$. Since $V''(0) = 0$, any family of perturbed Hamiltonians H_2^ε has a singularity $(x, y) = (x_0(\alpha, \beta, \varepsilon), 0)$ for all $(\alpha, \beta) \in \mathbb{R}^2$ and ε sufficiently small, with $x_0(\alpha, 0, 0) = 0$ for all $\alpha \in \mathbb{R}$. The proof is completely similar to that of Lemma 11. Since every perturbed system H_2^ε has a singularity that is mapped to the origin by a small translation, $H_{R,1}$ and H_2 only need to be versal in the class of deformations that fix the singularity at the origin.

Let $\mathscr{E}$ denote the ring of germs at 0 of functions $f: (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, and let $\mathscr{M}$ be its maximal ideal, consisting of germs f with $f(0, 0) = 0$. Then $\mathscr{M}^2$

is the ideal of germs that vanish at zero together with their first derivatives. As in the previous proof, let $J(h) \subset \mathscr{E}$ be the Jacobian ideal of h. Then

$$\mathscr{M} \cdot J(h) + \mathbb{R} \left\{ \frac{\partial}{\partial \lambda} H_{R,1} \Big|_{\lambda=0} \right\} = \mathscr{M}^2$$

$$\mathscr{M} \cdot J(h) + \mathbb{R} \left\{ \frac{\partial}{\partial \alpha} H_2 \Big|_{\alpha=\beta=0}, \frac{\partial}{\partial \beta} H_2 \Big|_{\alpha=\beta=0} \right\} = \mathscr{M}^2.$$

According to the Malgrange–Mather Preparation Theorem, see [14, 21, 25], this is a sufficient (and necessary) condition for $H_{R,1}$ and H_2 to be versal in the class of deformations that fix the singularity at the origin. ∎

Proof of Theorem 10(2). Assume that $V^{(3)}(0) = 0$, $V'(0) \neq 0$ and $V^{(4)}(0) \neq 0$. Then we have to prove that a local model for H_2 around the bifurcation point $(x, y; \alpha, \beta) = (0, 0; V''(0), 0)$ is given by

$$H_{R,2}(x, y; \lambda) = \tfrac{1}{2} y^2 + \lambda x + x^3,$$

and that H_2 itself is a local model.

The Taylor series of H_2 around the bifurcation point is

$$H_2(x, y; \alpha, \beta) = \tfrac{1}{2} y^2 + V'(0)(V''(0) - \alpha)\, x + \tfrac{1}{2} V''(0)(V''(0) - \alpha)\, x^2 + \tfrac{1}{6} V^{(4)}(0)\, V'(0)\, x^3 + O(x^4, \beta).$$

Finding a local diffeomorphism $\phi: (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and an invertible reparameterization $\alpha \mapsto \bar{\alpha}$ such that

$$H_2(\phi(x), y; \alpha, \beta) = \tfrac{1}{2} y^2 + \bar{\alpha}(\alpha)\, x + x^3 + O(\bar{\alpha}(\alpha)\, x^3, \beta)$$

is somewhat more complicated in this case. We abbreviate $d = \frac{1}{6} V'(0)\, V^{(4)}(0)$, then $d \neq 0$ and

$$H_2(x, y; \alpha, \beta) = \tfrac{1}{2} y^2 + V'(0)(V''(0) - \alpha)\, x + \tfrac{1}{2} V''(0)(V''(0) - \alpha)\, x^2 + dx^3 + x^4 f(x; \alpha) + O(\beta),$$

for some function f. Let $\phi^{-1}(x) = x \sqrt[3]{d + xf(x)}$, and $\tilde{\alpha} = d^{-1/3} V'(0)(V''(0) - \alpha)$, then ϕ is well-defined for sufficiently small x, and

$$\tilde{H}_2(x, y; \tilde{\alpha}, \beta) := H_2(\phi(x), y; \alpha, \beta) = \tfrac{1}{2} y^2 + \tilde{\alpha} x + \tilde{\alpha} c(\tilde{\alpha})\, x^2 + x^3 + O(\tilde{\alpha} x^3, \beta),$$

for some function c.

We want to get rid of the x^2-term. Let $\bar{x} = x + \frac{1}{3}\tilde{\alpha}c(\tilde{\alpha})$, and $\bar{\alpha} = \tilde{\alpha}(1 - \frac{1}{3}\tilde{\alpha}c(\tilde{\alpha})^2)$, then

$$\bar{H}_2(\bar{x}, y; \bar{\alpha}, \beta) := \tilde{H}_2(x, y; \tilde{\alpha}, \beta) = \tfrac{1}{2}y^2 + \bar{\alpha}\bar{x} + \bar{x}^3 + O(\bar{\alpha}\bar{x}^3, \beta).$$

Note that the transformations $x \mapsto \bar{x}$ and $\tilde{\alpha} \mapsto \bar{\alpha}$ are invertible for $\tilde{\alpha}$ sufficiently small.

The Hamiltonians $\bar{H}_2$ and $H_{R,2}$ are deformations of $h(x, y) := \frac{1}{2}y^2 + x^3$. In the same way as in the previous proofs, the Malgrange–Mather Preparation Theorem implies that these deformations are versal in the class of all deformations of h. ▌

4.3. *Proof of Theorem* 12

Finally we prove Theorem 12, dealing with the perturbative cases where H_2 has spatio-temporal symmetry, but its perturbations are only temporally symmetric.

Proof of Theorem 12. As in Subsection 2.3.3, let $V^\varepsilon(x) = V(x) + \varepsilon W(x; \varepsilon)$ be a perturbation of V, and let H_2^ε be the corresponding perturbed Hamiltonian. Assume that $V''(0) \neq 0$, $V^{(4)}(0) \neq 0$ and $c = V''(0)\, W^{(3)}(0; 0) - V^{(4)}(0)\, W'(0; 0) \neq 0$. Then we have to prove that a local model for H_2 around the bifurcation point $(x, y; \alpha, \beta) = (0, 0; V''(0), 0)$ is given by

$$H_{R*}(x, y; \lambda, \mu) = \tfrac{1}{2}y^2 - \lambda x^2 + \mu x^3 + \sigma x^4,$$

where $\sigma = \operatorname{sign} V''(0)\, V^{(4)}(0)$, and that H_2^ε is also a local model.

According to Lemma 11, H_2^ε has a singularity at $(x, y) = (x_0(\alpha, \beta, \varepsilon), 0)$, for all $(\alpha, \beta) \in \mathbb{R}^2$ and ε sufficiently small. The Taylor series of H_2^ε around this singularity is

$$\begin{aligned} H_2^\varepsilon(x, y; \alpha, \beta) = {} & \tfrac{1}{2}y^2 + \tfrac{1}{2}V''(0)\,\tilde{\alpha}\tilde{x}^2 + \tfrac{1}{3}c\,\varepsilon\tilde{x}^3 + \tfrac{1}{8}V''(0)\,V^{(4)}(0)\,\tilde{x}^4 \\ & + O(\tilde{x}^6, \varepsilon\tilde{x}^4, \tilde{\alpha}\tilde{x}^4, \varepsilon^2\tilde{x}^2, \varepsilon\tilde{\alpha}\tilde{x}^2, \beta\tilde{x}^2), \end{aligned}$$

where $\tilde{x} = x - x_0(\alpha, \beta, \varepsilon)$ and $\tilde{\alpha} = V''(0) + \varepsilon W''(0; 0) - \alpha$, compare (7). Analogous to the proof of Theorem 9, there exists a local diffeomorphism $\phi\colon (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and an invertible reparameterization $(\alpha, \varepsilon) \mapsto (\bar{\alpha}, \bar{\varepsilon})$ such that

$$\begin{aligned} \bar{H}_2^{\bar{\varepsilon}}(x, y; \bar{\alpha}, \beta) & := H_2^\varepsilon(\phi(x), y; \alpha, \beta) \\ & = \tfrac{1}{2}y^2 - \bar{\alpha}(\alpha, \varepsilon)\, x^2 + \bar{\varepsilon}(\alpha, \varepsilon)\, x^3 + \sigma x^4 + O(\bar{\alpha}x^4, \bar{\varepsilon}x^4, \beta). \end{aligned}$$

The Hamiltonians $\bar{H}_2^\varepsilon$ and H_{R*} are deformations of $h(x, y) := \frac{1}{2}y^2 + \sigma x^4$. By lemma 11, every perturbed system H_2^ε has a singularity (for all parameter values) that is mapped to the origin by a small translation. Therefore H_{R*}

and $\bar{H}_2^{\varepsilon}$ only need to be versal in the class of deformations that fix the singularity at the origin. The proof is completed by applying the Malgrange–Mather Preparation Theorem, as before. ▮

ACKNOWLEDGMENTS

The authors thank G. A. Lunter, K. Meyer, F. Takens, and F. O. O. Wagener for valuable discussion during the preparation of this paper.

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